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Energy momentum flows for the massive vector field

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Abstract. We present a causal trajectory interpretation for the massive vector field, based on the flows of rest energy and a conserved density defined using the time-like eigenvectors and eigenvalues of the stress-energy-momentum tensor. This work extends our previous work which used a similar procedure for the scalar field. The massive, spin-one, complex vector field is discussed in detail and solutions are classified using the Pauli-Lubanski spin vector. The flows of energy-momentum are illustrated in a simple example of standing waves in a plane.

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1. Introduction

In previous papers we have given a general method of constructing a causal trajectory interpretation, in the relativistic domain, for quantum mechanics [2], and in detail for the Klein-Gordon equation. This method has then been extended to the many-particle case (fixed number), which entailed the introduction of a Lorentz invariant rule for the coordination of the space time points on the individual particle trajectories which does not entail contradiction between Lorentz invariance and non-locality [3],[4].

In the classical context of general relativity Edelen [1] has given a natural definition of the flow of rest-energy and a conserved density in terms of the time-like eigenvectors and eigenvalues of the stress-energy-momentum tensor. In this paper we extend the detailed discussion to the massive, spin 1, complex vector field and give an example of standing waves on a space-like plane.

In sections two and three we give a general account of the massive vector field formalism. The classification of the solutions by the Pauli-Lubanski spin vector is then detailed in sections four, five and six, which enables a covariant characterisation of intrinsic spin. The symmetric stress-energy-momentum tensor and its eigenvectors and eigenvalues are given in sections seven and eight which requires an extension from the massless maxwellian case. The two parts of the complex field are shown to give rise to a tetrad of vectors which each define a time-like two-plane and space-like two-plane.

2. The massive vector field

The massive vector, complex field $\phi^\mu(x)$ has a Lagrangian density

$$\begin{aligned}\mathcal{L} = \frac{1}{2}\bar{\mathcal{G}}_{\mu\nu}\mathcal{G}^{\mu\nu} - \frac{1}{2}\bar{\mathcal{G}}_{\mu\nu}(\partial^\mu\phi^\nu - \partial^\nu\phi^\mu) \\ - \frac{1}{2}(\partial^\mu\bar{\phi}^\nu - \partial^\nu\bar{\phi}^\mu)\mathcal{G}_{\mu\nu} \\ + m^2\bar{\phi}_\mu\phi^\mu \end{aligned} \tag{1}$$

with the derived field equations

$$\mathcal{G}^{\mu\nu} = \partial^\mu\phi^\nu - \partial^\nu\phi^\mu \tag{2}$$

$$-\partial_\nu\mathcal{G}^{\mu\nu} + m^2\phi^\mu = 0 \tag{3}$$

These are the Proca equations for the vector fields, equivalent to

$$(\Box + m^2)\phi_\mu = 0 \tag{4}$$

$$\partial_\mu\phi^\mu = 0 \tag{5}$$

The condition in equation (5) is, of course, essential to obtain three independent components of ϕ^μ in a covariant way, so as to describe a spin one field.

It is sometimes convenient to introduce an auxiliary condition to ensure the splitting of an arbitrary solution ϕ^μ into a transverse (spin 1) part and a scalar part (as for

Stueckelberg's Lagrangian). One adds a term $-\lambda(\partial\cdot\phi)(\partial\cdot\bar{\phi})$ to $\mathcal{L}$. The field ϕ^T_μ is then divergenceless, where

$$\phi^T_\mu = \phi_\mu + \frac{\lambda}{m^2}\partial_\mu(\partial\cdot\phi) \tag{6}$$

The equations of motion will be

$$(\Box + m^2)\phi_\mu - (1-\lambda)\partial_\mu(\partial\cdot\phi) = 0 \tag{7}$$

from which, taking the divergence of both sides one gets

$$[\lambda\Box + m^2](\partial\cdot\phi) = 0 \tag{8}$$

One can then explore, in the usual way, the limits $m \to 0$, and $\lambda \to 0$.

For the real vector field and field quantization procedures see [5]

The field $\phi_\mu(x)$ will, as a result of the condition given in equation (5) have Fourier components of the form

$$a^\mu(k)e^{-ikx} \tag{9}$$

with

$$k_\mu a^\mu = 0 \tag{10}$$

that is a^μ is space-like and k_μ is time-like. In a general wave packet, however, ϕ_μ need not be space-like.

In general one can write [5]

$$\phi_\mu(x) = \int \frac{d^3k}{2k_0(2\pi)^3}\Sigma^3_{\lambda=1}[a^{(\lambda)}(k)\epsilon^{(\lambda)}_\mu(k)e^{-ikx} + a^{(-\lambda)}(k)\epsilon^{(-\lambda)}_\mu(k)e^{+ikx}] \tag{11}$$

where $k_0 = \sqrt{(k^2 + m^2)}$.

The three space-like, orthonormalised vectors $\epsilon^{(\lambda)}_\mu(k)$ are orthogonal to the time-like vector k_μ and

$$\epsilon^{(\lambda)}(k)\cdot\epsilon^{(\lambda')}(k) = \delta_{\lambda\lambda'} \tag{12}$$

$$\Sigma_\lambda\epsilon^{(\lambda)}_\mu(k)\epsilon^{(\lambda)}_\nu(k) = -\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}\right] \tag{13}$$

3. Electromagnetic Interaction

In order to include the effects of interaction of a charged massive vector field with an electromagnetic field $F_{\mu\nu}$ with vector potential A_μ one introduces the gauge covariant operator

$$D_\mu = \partial_\mu + ieA_\mu \tag{14}$$

with

$$[D_\mu, D_\nu] = ieF_{\mu\nu} \tag{15}$$

The equations of motion will be

$$-D_\nu\mathcal{G}^{\mu\nu} + m^2\phi^\mu = 0 \tag{16}$$

with

$$\mathcal{G}^{\mu\nu} = D^\mu \phi^\nu - D^\nu \phi^\mu \tag{17}$$

The second order wave equation and the new divergence condition imposed on ϕ^μ require more detailed derivation.

$$D_\mu(D_\nu \mathcal{G}^{\mu\nu}) = m^2 D_\mu \phi^\mu \tag{18}$$

but

$$D_\mu D_\nu = D_\nu D_\mu + ieF_{\mu\nu} \tag{19}$$

hence

$$\begin{aligned} D_\mu(D_\nu \mathcal{G}^{\mu\nu}) &= D_\nu(D_\mu \mathcal{G}^{\mu\nu}) + ieF_{\mu\nu}\mathcal{G}^{\mu\nu} \\ &= D_\nu(-m^2\phi^\nu) + ieF_{\mu\nu}\mathcal{G}^{\mu\nu} \\ &= m^2(D_\nu \phi^\nu) \end{aligned}$$

Therefore

$$D_\nu \phi^\nu = \frac{ie}{2m^2} F_{\mu\nu}\mathcal{G}^{\mu\nu} \tag{20}$$

This last condition, equation (20), is the new divergence condition.

The second order equation is given by

$$\begin{aligned} -D_\nu \mathcal{G}^{\mu\nu} &= D_\nu(D^\nu \phi^\mu) - D_\nu(D^\mu \phi^\nu) \\ &= (D_\nu D^\nu)\phi^\mu - D^\mu(D_\nu \phi^\nu) + ieF^\mu_\nu \phi^\nu \end{aligned}$$

Therefore

$$(D_\nu D^\nu)\phi^\mu + m^2\phi^\mu + ieF^\mu_\nu \phi^\nu - D^\mu(D_\nu \phi^\nu) = 0 \tag{21}$$

and so

$$(D_\nu D^\nu)\phi^\mu + m^2\phi^\mu + ieF^\mu_\nu \phi^\nu - \frac{ie}{2m^2} D^\mu(F_{\alpha\beta}\mathcal{G}^{\alpha\beta}) = 0 \tag{22}$$

It is of interest to compare the term $ieF^\mu_\nu \phi^\nu$ with the term that occurs in the second order Dirac equation [5] namely, $-g\frac{e}{2}[i\underline{\alpha} \cdot \underline{E} + \underline{\sigma} \cdot \underline{B}]$, in the usual notation.

F^μ_ν can be expressed in terms of the infinitesimal generators for spatial rotations and Lorentz boosts [6]

$$\underline{M} = [M_{32}, M_{13}, M_{21}] \tag{23}$$

$$\underline{N} = [M_{01}, M_{02}, M_{03}] \tag{24}$$

$M_{\mu\nu}$ being an anti symmetric tensor corresponding to infinitesimal rotations in the $\mu\nu$ plane (from μ to ν)

Explicitly

$$M_1 = M_{32} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{bmatrix}$$

$$M_2 = M_{13} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$M_3 = M_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N_1 = M_{01} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N_2 = M_{02} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N_3 = M_{03} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Since

$$[F^\mu_\nu] = \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{bmatrix}$$

one can write

$$F^\mu_\nu = N_k E^k - M_k B^k$$

That is

$$F^\mu_\nu \phi^\nu = e[(iN_k) \cdot E^k - (iM_k) B^k]\phi^\nu$$

This term compares directly with that in the Dirac equation as given above.

4. Classification of Solutions according to the Pauli-Lubanski Spin Vector W^μ

A covariant characterization of intrinsic spin can be given in terms of the Pauli-Lubanski operator W_μ

$$W_\mu = -\frac{1}{2} i\epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma$$

[5]. Using the previously defined infinitesimal rotation operators M_k, N_k one has ‡

$$\begin{aligned}
W^1 &= -iP_0M^1 - i(P_2N^3 - P_3N^2)\\
W^2 &= -iP_0M^2 - i(P_3N^1 - P_1N^3)\\
W^3 &= -iP_0M^3 - i(P_1N^2 - P_2N^1)\\
W^0 &= -iP_kM^k
\end{aligned} \tag{25}$$

where P_μ is the usual covariant momentum operator.

The actual hermitian spin operators $\underline{S}$ are not covariant and take the form [8]

$$\underline{S} = \frac{1}{m}[\underline{W} - \frac{W_0\underline{P}}{m+P_0}]$$

and obey the commutation rule

$$[S_p, S_q] = i\epsilon_{pqr}S_r$$

and $[S_r, P_\lambda] = 0$ as expected for spin operators.

We consider the eigenfunctions of W^3 and W^0. In the rest frame for a single plane wave there will, of course, be no difference between $\underline{S}$ and $\underline{W}$.

5. Eigenfunctions of W^3

Consider a single plane wave

$$\epsilon_\mu(k)e^{-ik\cdot x}$$

with

$$W^3\begin{pmatrix}\epsilon_0\\ \epsilon_1\\ \epsilon_2\\ \epsilon_3\end{pmatrix}e^{-ik\cdot x} = \lambda\begin{pmatrix}\epsilon_0\\ \epsilon_1\\ \epsilon_2\\ \epsilon_3\end{pmatrix}e^{-ik\cdot x}$$

Explicitly

$$\begin{aligned}
-ik_2\epsilon_1 + ik_1\epsilon_2 &= \lambda\epsilon_0\\
-ik_2\epsilon_0 + ik_0\epsilon_2 &= \lambda\epsilon_1\\
+ik_1\epsilon_0 - ik_0\epsilon_1 &= \lambda\epsilon_2\\
0 &= \lambda\epsilon_3
\end{aligned}$$

for which $\lambda = 0, \lambda = \pm\sqrt{k_0^2 - k_1^2 - k_2^2}$

Case $\lambda = 0$

$\epsilon_1 = k_1, \epsilon_2 = k_2, \epsilon_0 = k_0$

ϵ_3 is chosen to give

$$\partial^\mu\epsilon_\mu e^{-ik\cdot x} = 0$$

‡ The factor i is sometimes absorbed into M_k, N_k to give a hermitian and anti-hermitian operator.

leading to

$$\epsilon_3 = \frac{m^2 + k_3^2}{k_3}$$

$$\begin{pmatrix} \epsilon_0 \\ \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} e^{-ik\cdot x} = \begin{pmatrix} k_0 \\ k_1 \\ k_2 \\ \frac{m^2+k_3^2}{k_3} \end{pmatrix} e^{-ik\cdot x}$$

Case $\lambda = \pm\sqrt{k_0^2 - k_1^2 - k_2^2}$

$$\frac{\epsilon_1}{\epsilon_0} = \frac{\lambda k_0 - ik_1k_2}{\lambda k_1 - ik_0k_2}$$

$$\frac{\epsilon_2}{\epsilon_0} = \frac{\lambda k_0 + ik_1k_2}{\lambda k_2 + ik_0k_1}$$

For the special circumstance in which only k_0 and k_3 are non-zero one has the two cases

Case $\lambda = 0$

$$\begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} k_0 \\ 0 \\ 0 \\ \frac{k_0^2}{k_3} \end{pmatrix} e^{-i(k_0x^0+k_3x^3)}$$

which is the case of longitudinal polarization.

Case $\lambda = \pm k_0$

$$\begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} e^{-i(k_0x^0+k_3x^3)}$$

which corresponds with the case of right and left circular polarization.

5.1. Standing waves with spin perpendicular to the direction of propagation

For the case in which only k_0 and k_1 are non-zero one has

$$\frac{\epsilon_1}{\epsilon_0} = \frac{k_0}{k_1}$$

$$\frac{\epsilon_2}{\epsilon_0} = -i\frac{\lambda}{k_1}$$

and

$\epsilon_0 = k_1, \epsilon_1 = k_0, \epsilon_2 = \mp|\lambda| = \mp im$

For the case in which only k_0 and k_2 are non-zero one has

$\epsilon_0 = k_2, \epsilon_2 = k_0, \epsilon_1 = \pm im$

In both cases

$$\partial^\mu \epsilon_\mu e^{-ik\cdot x} = 0$$

One can then construct the single frequency state for counter propagating waves in the x^1 and x^2 directions with spin in the third x^3 direction. In this case there is no difference between S^3 and $\frac{W^3}{m}$. In the positive spin case one has

$$\begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} -i(k_1 \sin(k_1 x_1) + k_2 \sin(k_2 x_2)) \\ im\cos(k_2 x_2) + k_0 \cos(k_1 x_1) \\ -im\cos(k_1 x_1) + k_0 \cos(k_2 x_2) \\ 0 \end{pmatrix} e^{-i(k_0 x^0)} \tag{26}$$

with $|k_1| = |k_2|$.

6. Eigenfunctions of W^0

As for W^3

$$W^0 \begin{pmatrix} \epsilon_0 \\ \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} e^{-ik\cdot x} = \lambda \begin{pmatrix} \epsilon_0 \\ \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} e^{-ik\cdot x}$$

Explicitly

$$\begin{aligned} -ik_3\epsilon_2 + ik_2\epsilon_3 &= \lambda\epsilon_1 \\ +ik_3\epsilon_1 - ik_1\epsilon_3 &= \lambda\epsilon_2 \\ -ik_2\epsilon_1 + ik_1\epsilon_2 &= \lambda\epsilon_3 \end{aligned}$$

which gives

$$\begin{aligned} &\lambda = 0 \\ &\lambda = \pm\sqrt{k_0^2 - m^2} = \pm\sqrt{k_1^2 + k_2^2 + k_3^2} \end{aligned}$$

since

$$i(\underline{k} \times \underline{\epsilon}) = \lambda \underline{\epsilon} \tag{27}$$

then

$$(\underline{k} \times \underline{\epsilon})(\underline{k} \times \underline{\bar{\epsilon}}) = \lambda^2 \underline{\epsilon} \cdot \underline{\bar{\epsilon}} \tag{28}$$

and

$$\lambda^2 \underline{\epsilon} \cdot \underline{\bar{\epsilon}} = (\underline{k} \cdot \underline{k})(\underline{\epsilon} \cdot \underline{\bar{\epsilon}}) - (\underline{k} \cdot \underline{\epsilon})(\underline{k} \cdot \underline{\bar{\epsilon}}) \tag{29}$$

But, from equation (27)

$$\lambda(\underline{k} \cdot \underline{\epsilon}) = i\underline{k} \cdot (\underline{k} \times \underline{\epsilon}) = 0 \tag{30}$$

Hence $\underline{\epsilon}$ is transverse with respect to $\underline{k}$, with $\lambda^2 = \underline{k} \cdot \underline{k}$ from equation (29) unless $(\underline{k} \times \underline{\epsilon} = 0)$. Some simple algebra gives, for $\lambda = 0$,

$$\begin{pmatrix} \epsilon_0 \\ \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} = \begin{pmatrix} \frac{k_0^2 - m^2}{k_0} \\ k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

and for $\lambda = \pm\sqrt{\underline{k} \cdot \underline{k}}$, from equation (27) one easily gets

$$\frac{\epsilon_1}{\epsilon_3} = \frac{-k_1 k_3 + i\lambda k_2}{\lambda^2 - k_3^2}$$

$$\frac{\epsilon_2}{\epsilon_3} = \frac{-k_2 k_3 - i\lambda k_1}{\lambda^2 - k_3^2}$$

$$\epsilon_0 = 0$$

7. Symmetric stress energy-momentum tensor

A symmetric tensor can be derived by considering the variation of the Lagrangian with respect to the metric tensor $g_{\mu\nu}$.

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \left[\frac{\partial \sqrt{-g}\mathcal{L}}{\partial g^{\mu\nu}} - \partial^\alpha \left(\frac{\partial \sqrt{-g}\mathcal{L}}{\partial \left(\partial^\alpha g^{\mu\nu}\right)} \right) \right]$$

In the case being considered there are no terms involving $\partial^\alpha g^{\mu\nu}$ so that one finds

$$T_{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \mathcal{L} g_{\mu\nu}$$

Inserting the metric tensor into the previously given $\mathcal{L}$ one gets

$$\begin{aligned} T_{\mu\nu} = {} & \bar{\mathcal{G}}_{\mu\alpha}\mathcal{G}_\nu^\alpha + \mathcal{G}_{\mu\alpha}\bar{\mathcal{G}}_\nu^\alpha \\ & + m^2(\bar{\phi}_\mu \phi_\nu + \phi_\mu \bar{\phi}_\nu) \\ & + g_{\mu\nu} \left(\frac{1}{2} \left(\bar{\mathcal{G}}_{\alpha\beta} \mathcal{G}^{\alpha\beta} \right) - m^2 \bar{\phi}^\alpha \phi_\alpha \right) \end{aligned} \tag{31}$$

Expressing the complex field in terms of two real fields $\phi_\mu(1)$ and $\phi_\mu(2)$

$$\phi_\mu = \phi_\mu(1) + i\phi_\mu(2)$$

$T_{\mu\nu}$ becomes the sum of the stress-energy momentum tensors $T_{\mu\nu}(1)$ and $T_{\mu\nu}(2)$ of the above two real fields.

$$\begin{aligned} \frac{1}{2} T_{\mu\nu}(1) = {} & G_{\mu\alpha}(1) G_\nu^\alpha(1) + m^2 \phi_\mu(1) \phi_\nu(1) \\ & + g_{\mu\nu} \left[\frac{1}{4} \left(G^{\alpha\beta}(1) G_{\alpha\beta}(1) \right) - \frac{m^2}{2} \phi^\alpha(1) \phi_\alpha(1) \right] \end{aligned} \tag{32}$$

and similarly for $\frac{1}{2}T_{\mu\nu}(2)$. Where G is a purely real antisymmetric field tensor, $\mathcal{G}$ is reserved for the complex field tensor.

The parts of $T_{\mu\nu}(1)$ and $T_{\mu\nu}(2)$ not depending explicitly on the mass have the same form as for the massless real electromagnetic field.

It is known for the massless real (spin 1) field that $T_{\mu\nu}$ can be expressed in terms of a null tetrad of vectors which, however, are not unique. A natural choice is given by the null eigenvectors of $G_{\mu\nu}$ (for each of the two real fields) [9],[10]. Another way of specifying the null tetrad for the massless real electromagnetic field has been given by Misner et al [11] and developed in a recent paper by Garat [13]. We give our version of this latter method, useful in our case, in the appendix to this paper.

The eigenvector equation for either of the two *real* fields is of the form

$$G_{\mu\nu}X^{\nu} = \lambda X_{\mu}$$

(The "inscript" 1,2 is omitted). It then follows from the antisymmetry of $G_{\mu\nu}$ that the solutions fall into two groups.

(i) Null eigenvectors defining a time-like two-plane with equal real eigenvalues of opposite sign.

(ii) Null eigenvectors defining a space-like two plane with equal pure imaginary eigenvalues of opposite sign

We denote the null tetrad by $l^{\mu}, n^{\nu}, m^{\mu}, \bar{m}^{\mu}$ using the Penrose notation [7]

$$\begin{aligned} &l^{\mu}l_{\mu} = n^{\mu}n_{\mu} = m^{\mu}m_{\mu} = \bar{m}_{\mu}\bar{m}^{\mu} = 0 \\ &l_{\mu}n^{\mu} = 1 \\ &m_{\mu}\bar{m}^{\mu} = -1 \end{aligned} \tag{33}$$

and all other scalar products vanish.

The metric is given by

$$g_{\mu}^{\nu} = n_{\mu}l^{\nu} + l_{\mu}n^{\nu} - \bar{m}_{\mu}m^{\nu} - m_{\mu}\bar{m}^{\nu}$$

with the signature $(+,-,-,-)$. Then

$$G_{\mu\nu} = \lambda(1)[l_{\mu}n_{\nu} - n_{\mu}l_{\nu}] + i\lambda(2)[m_{\mu}\bar{m}_{\nu} - \bar{m}_{\mu}m_{\nu}]$$

It is easily shown that

$$\begin{aligned} &\frac{1}{2}G_{\alpha\beta}G^{\alpha\beta} = -(\lambda^2(1) - \lambda^2(2)) \\ &G_{\mu\alpha}G_{\nu}^{\alpha} = \lambda^2(1)[l_{\mu}n_{\nu} + l_{\nu}n_{\mu}] + \lambda^2(2)[m_{\mu}\bar{m}_{\nu} + m_{\nu}\bar{m}_{\mu}] \end{aligned}$$

or, using the metric tensor

$$G_{\mu\alpha}G_{\nu}^{\alpha} = \left(\lambda^2(1) + \lambda^2(2)\right)[l_{\mu}n_{\nu} + l_{\nu}n_{\mu}] - \lambda^2(2)g_{\mu\nu}$$

with

$$\left(\lambda^2(1) + \lambda^2(2)\right) = \frac{1}{4}\left[\left(G_{\alpha\beta}G^{\alpha\beta}\right)^2 + \left(G^{*}_{\alpha\beta}G^{\alpha\beta}\right)^2\right]^{\frac{1}{2}}$$

Each real field gives expressions of the same form. However, in general, the two time-like two-planes defined by l_{μ}, n_{ν} will be different.

A useful method in finding the eigenfunctions of the total $T_{\mu\nu}$ is to construct a common time-like two-plane. A unique prescription can be given following a discussion in [7] of the properties of the Lorentz transformations. Given any four, distinct, real null vectors one has a unique time-like two-plane Ω which contains

(i) One vector from each of the planes defined by $(l_\mu(1), n_\mu(1))$ and $(l_\mu(2), n_\mu(2))$.

(ii) One normal to each of the same two-planes.

If Ω contains the orthogonal vectors $\underline{t}, \underline{z}$, then the restricted Lorentz transformation relating the two-planes and conserving Ω is a rotation about $\underline{z}$ and a boost along the $\underline{z}$ axis. Once the tetrads for the two real fields have been found it is straightforward to determine the overall eigenvectors.

8. Classification of the eigenvectors of the real and imaginary parts of the stress energy-momentum tensor

$$
\begin{aligned}
\frac{T_{\mu\nu}}{2} &= G_{\mu\alpha}G^{\alpha}_{\nu} + m^2\phi_\mu\phi_\nu \\
&\quad + g_{\mu\nu}\left[\frac{1}{4}\left(G^{\alpha\beta}G_{\alpha\beta}\right) - \frac{m^2}{2}\phi^\alpha\phi_\alpha\right]
\end{aligned} \tag{34}
$$

It is convenient to switch to a tetrad in terms of space and time components

$$
\begin{aligned}
l_\mu &= \frac{1}{\sqrt{2}}(T_\mu + Z_\mu) \\
n_\mu &= \frac{1}{\sqrt{2}}(T_\mu - Z_\mu) \\
m_\mu &= \frac{1}{\sqrt{2}}(X_\mu - iY_\mu) \\
\bar{m}_\mu &= \frac{1}{\sqrt{2}}(X_\mu + iY_\mu)
\end{aligned} \tag{35}
$$

As we are concerned here with the time-like or space-like character of the eigenvectors we may just consider a tensor of the form

$$
k[(T_\mu T_\nu - Z_\mu Z_\nu) + (X_\mu X_\mu + Y_\mu Y_\mu)] + m^2\phi_\mu\phi_\nu
$$

where $k = (\lambda^2(1) + \lambda^2(2))$ for brevity. The first part of $T_{\mu\nu}$, for either of the real fields is unaltered in form when the tetrad is "steered" preserving the time-like two-plane and the space-like two-plane. Therefore, *without loss of generality*, one may assume that ϕ_μ takes one of two forms §. The algebraic details are given in the appendix with the specification of the tetrads.

(i) $\phi_\mu = \alpha T_\mu + \gamma X_\mu$

(ii) $\phi_\mu = \beta Z_\mu + \delta Y_\mu$

Taking an eigenvector E_μ in the form

$$
E_\mu = aT_\mu + bZ_\mu + cX_\mu + dY_\mu
$$

§ It is necessary to impose $\partial \cdot \phi = 0$ for solutions of the Proca equation.

one gets, denoting the eigenvalue by K, two cases.

For case (i)

$$
\begin{aligned}
Ka &= ka + m^2\alpha(\phi \cdot E) \\
Kb &= kb \\
Kc &= -kc + m^2\gamma(\phi \cdot E) \\
Kd &= -kd
\end{aligned}
\tag{36}
$$

and

$$\phi \cdot E = a\alpha - c\gamma$$

From the equations involving a and c only one gets

$$\left(\frac{K-k}{K+k}\right)\frac{a}{c} = \frac{\alpha}{\gamma} \tag{37}$$

One may take $a = \alpha(K+k)$ and $c = \gamma(K-k)$. The eigenvalue equation is

$$(K - k - m^2\alpha^2)(K + k + m^2\gamma^2) + (m^2\alpha\gamma)^2 = 0 \tag{38}$$

The nature of the eigenvectors in this case is given by their norm squared

$$a^2 - c^2 = \alpha^2(K+k) - \gamma^2(K-k) = (K^2 - k^2)\frac{1}{m^2}$$

the last equality following from the equations (37, 38). Since $(K - k - m^2\alpha^2)(K + k + m^2\gamma^2)$ is negative one finds that

- If K is positive then $K < k + m^2\alpha^2$.
- If K is negative then $K < -(k + m^2\gamma^2)$.

In this latter case $a^2 - c^2$ is positive and E_μ will be time-like. Since the two solutions are orthogonal the other solution will be space-like. There are two further space-like solutions with $\phi \cdot E = 0$

- b only $c = d = 0$, $K = +k$
- d only $c = d = 0$, $K = -k$

One has therefore four orthogonal solutions

$$E_\mu = \alpha(K+k)T_\mu + \gamma(K-k)X_\mu$$

(two solutions), and

$$
\begin{aligned}
E_\mu &= Z_\mu \\
E_\mu &= Y_\mu
\end{aligned}
\tag{39}
$$

For case (ii)

$$\begin{aligned} Ka &= ka \\ Kb &= kb + m^2\beta(\phi \cdot E) \\ Kc &= -kc \\ Kd &= -kd + m^2\delta(\phi \cdot E) \end{aligned}$$

and

$$\phi \cdot E = -(b\beta + d\delta)$$

One immediately finds one time-like solution, with $K = +k$

$$E_\mu = aT_\mu$$

One space-like solution, with $K = -k$

$$E_\mu = cX_\mu$$

and two space-like solutions (orthogonal to the first two solutions).

$$\left(\frac{K-k}{K+k}\right)\frac{b}{d} = \frac{\beta}{\delta} \tag{40}$$

One may take $b = \beta(K+k)$ and $d = \delta(K-k)$. The eigenvalue equation is

$$(K - k + m^2\beta^2)(K + k + m^2\delta^2) - (m^2\beta\delta)^2 = 0 \tag{41}$$

Again one has therefore four orthogonal solutions

$$E_\mu = \beta(K+k)Z_\mu + \delta(K-k)Y_\mu$$

(two solutions), and

$$\begin{aligned} E_\mu &= T_\mu \\ E_\mu &= X_\mu \end{aligned} \tag{42}$$

The overall eigenvalues of $\frac{T_{\mu\nu}}{2}$ from equation (34) will be

$$\lambda = K - \frac{m^2}{2}\phi^\alpha\phi_\alpha$$

For case (i)

$$\lambda = \pm\sqrt{k^2 + km^2(\alpha^2 + \gamma^2) + (\frac{1}{2}m^2\phi^\alpha\phi_\alpha)^2}$$

and

$$\lambda = \pm k - \frac{m^2}{2}\phi^\alpha\phi_\alpha$$

For case (ii)

$$\lambda = \pm\sqrt{k^2 + km^2(\delta^2 - \beta^2) + (\frac{1}{2}m^2\phi^\alpha\phi_\alpha)^2}$$

and

$$\lambda = \pm k - \frac{m^2}{2}\phi^\alpha \phi_\alpha$$

where

$$\begin{aligned} k &= \frac{1}{4}\left[\left(G_{\alpha\beta}G^{\alpha\beta}\right)^2 + \left({}^*G_{\alpha\beta}G^{\alpha\beta}\right)^2\right]^{\frac{1}{2}} \\ &= \frac{1}{4}\left({}^{(\theta)}G_{ab}\right)\left({}^{(\theta)}G^{ab}\right) \end{aligned} \tag{43}$$

where θ is the extremal angle of duality rotation (see appendix). The eigenvectors of $T_{\mu\nu}$, including the mass term, are as follows.

Case (i): The maxwellian part of $T_{\mu\nu}$ gives a time-like two-plane $\left(\widehat{T} - \widehat{Z}\right)$ and the potential ϕ_μ is in the plane of $\left(\widehat{T} - \widehat{X}\right)$, as shown in figure 1. ϕ_μ can be either time-like or space-like.

The time-like two-plane $\left(\widehat{T} - \widehat{X}\right)$ contains two eigenvectors which are Lorentz boosted $\widehat{T}$ and $\widehat{X}$ as can be easily seen from (40),(41) and (42).

Case (ii): Again the maxwellian part of $T_{\mu\nu}$ gives a time-like two-plane $\left(\widehat{T} - \widehat{Z}\right)$ but ϕ_μ is in the plane of $\left(\widehat{Z} - \widehat{Y}\right)$, as shown in figure 2. In this case E_μ (time-like) is given by $\widehat{T}$ and $\widehat{\phi}$ is perpendicular to $\widehat{T}$.

In both cases a definite time-like eigenvector is given; either $\widehat{T}$ or a Lorentz-boosted version, the tetrads being given in the appendix.

9. Complex field and eigenvectors of $T_{\mu\nu}$

In both cases the eigenvectors of $T_{\mu\nu}(i)$ enable one to define a time-like two-plane. Just considering the parts of $T_{\mu\nu}(i)$ defining the two time-like two-planes (omitting the terms in the mass) one has

$$\begin{aligned} T_{\mu\nu} = 2k_1 &\left[T_\mu(1)T_\nu(1) - Z_\mu(1)Z_\mu(1)\right] \\ &+ 2k_2\left[T_\mu(2)T_\nu(2) - Z_\mu(2)Z_\mu(2)\right] \end{aligned} \tag{44}$$

The $Z_{(1,2)}$ direction is a purely conventional label. The common, unique, plane Ω with axes $\hat{t}, \hat{z}$ is such that [7]

(i) $\widehat{Z}$ is orthogonal to $\widehat{Z}(1)$,$\widehat{Z}(2)$

(ii) $T_\mu(i) = \cosh(\theta_i)t_\mu + \sinh(\theta_i)z_\mu$

The time-like eigenvector of the $T_{\mu\nu}$ given in (44) will necessarily lie in the plane Ω and take the form

$$T_\mu = \cosh(\theta)t_\mu + \sinh(\theta)z_\mu$$

Ω is unique but, of course, $\hat{t}$ and $\hat{z}$ may be subjected to a Lorentz boost.

There also exists a space-like eigenvector in the Ω plane of the form:

$$\sinh(\theta)t_\mu + \cosh(\theta)z_\mu$$

The eigenvalues given by

$$T_{\mu\nu}E^{\nu} = \Lambda E_{\mu}$$

are easily derived and are

(i) Time-like eigenvector

$$\Lambda = (k_1 + k_2) + \left[(k_1 \cosh(2\theta_1) + k_2 \cosh(2\theta_2))^2 - (k_1 \sinh(2\theta_1) + k_2 \sinh(2\theta_2))^2\right]^{\frac{1}{2}}$$

(ii) Space-like eigenvector

$$\Lambda = (k_1 + k_2) - \left[(k_1 \cosh(2\theta_1) + k_2 \cosh(2\theta_2))^2 - (k_1 \sinh(2\theta_1) + k_2 \sinh(2\theta_2))^2\right]^{\frac{1}{2}}$$

The three velocity derived from the time-like eigenvector is given by $\tanh(\theta)$

$$\tanh(\theta) = \frac{[(k_1 \cosh(2\theta_1) + k_2 \cosh(2\theta_2))] - [\Lambda - (k_1 + k_2)]}{[(k_1 \cosh(2\theta_1) + k_2 \cosh(2\theta_2))] + [\Lambda - (k_1 + k_2)]}$$

One can note that, in the massless case, the above results now give a definite time-like eigenvector in the complex case.The terms involving $Z_\mu(1)$ and $Z_\mu(2)$ can be similarly diagionalised. $Z_\mu(1)$ and $Z_\mu(2)$ define a plane orthogonal to $\widehat{Z}$ so one can use definite unit orthogonal vectors $\widehat{X}, \widehat{Y}$ in this plane (in general $k_1 \neq k_2$).

An algebraic simplification can be brought about by applying a lorentz boost to $(\hat{t}, \hat{z})$ so that

$$\theta'_1 = -\theta'_2 = \frac{\theta'_1 - \theta'_2}{2} = \frac{\theta_1 - \theta_2}{2}$$

and then

$$\tanh(2\theta') = \left(\frac{k_1 - k_2}{k_1 + k_2}\right) \tanh(\theta_1 - \theta_2)$$

and

$$T_\mu(1)T^\mu(2) = \cosh(\theta_1 - \theta_2)$$

(the dashed variables refer to the boosted axes). In the massive vector field case the extra mass term, apart from the terms in $g_{\mu\nu}$, will be

$$m^2 \left(\phi_\mu(1)\phi_\nu(1) + \phi_\mu(2)\phi_\nu(2)\right)$$

In all cases mentioned in section (4) there will now be cross terms between the $\left(\hat{t} - \hat{z}\right)$ plane and the $\left(\hat{X} - \hat{Y}\right)$ plane defined for the massless case.

One is now left with a complicated algebraic problem to find the common time-like eigenvector of the total $T_{\mu\nu}$ which is left for future investigations.

10. An illustrative example

We illustrate the flows of energy momentum using standing waves in the $x_1 - x_2$ plane and with spin W^3 equal to $+m$‖. We have previously calculated the vector potential for this case in section (5.1) which gave for the real and imaginary parts of $G_{\mu\nu}$, $[G_{\mu\nu}]_R$ and $[G_{\mu\nu}]_I$ respectively

$$[G_{\mu\nu}]_R = \begin{bmatrix} 0 & mk_0c_2 & -mk_0c_1 & 0 \\ -mk_0c_2 & 0 & 0 & 0 \\ mk_0c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[G_{\mu\nu}]_I = \begin{bmatrix} 0 & -(k_0^2 - k_1^2)c_1 & (k_0^2 - k_2^2)c_2 & 0 \\ (k_0^2 - k_1^2)c_1 & 0 & m(k_1s_1 + k_2s_2) & 0 \\ (k_0^2 - k_2^2)c_2 & m(k_1s_1 + k_2s_2) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $c_2 = cos(k_2x_2), c_1 = cos(k_1x_1), s_2 = sin(k_2x_2)$ and $s_1 = sin(k_1x_1)$. The null eigenvectors are easily calculated. For the real field they are

$$\begin{pmatrix} 1 \\ \pm\frac{\cos(k_2x_2)}{\sqrt{\cos^2(k_2x_2)+\cos^2(k_1x_1)}} \\ \mp\frac{\cos(k_1x_1)}{\sqrt{\cos^2(k_2x_2)+\cos^2(k_1x_1)}} \\ 0 \end{pmatrix}$$

This null dyad gives the time-like unit vector

$$\widehat{T}_R = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{45}$$

and the space-like unit vector

$$\widehat{X}_R = \begin{pmatrix} 0 \\ \frac{\cos(k_2x_2)}{\sqrt{\cos^2(k_2x_2)+\cos^2(k_1x_1)}} \\ -\frac{\cos(k_1x_1)}{\sqrt{\cos^2(k_2x_2)+\cos^2(k_1x_1)}} \\ 0 \end{pmatrix} \tag{46}$$

For the imaginary field the null vector is

$$\begin{pmatrix} X^0 \\ X^1 \\ X^2 \\ 0 \end{pmatrix}$$

‖ In an earlier paper we used a similar example to calculate the particle trajectories according to the de Broglie-Bohm interpretation of the Dirac equation [12].

with

$$\frac{X^1}{X^0} = \frac{-(\lambda A - \alpha B)}{(\lambda^2 + \alpha^2)}$$

and

$$\frac{X^2}{X^0} = \frac{-(-\alpha A + \lambda B)}{(\lambda^2 + \alpha^2)}$$

where

$$\begin{aligned}
A &= m^2 \cos(k_1 x_1) \\
B &= -m^2 \cos(k_2 x_2) \\
\alpha &= m(k_1 \sin(k_1 x_1) + k_2 \sin(k_2 x_2)) \\
\lambda &= \pm\sqrt{(A^2 + B^2 - \alpha^2)} \qquad (47) \\
& \qquad (48)
\end{aligned}$$

which give normalized unit vectors for the time-like vector

$$\widehat{T}_I = \begin{pmatrix} \frac{m^2}{\lambda}\sqrt{\cos^2(k_1x_1) + \cos^2(k_2x_2)} \\ \frac{\alpha}{\lambda}\frac{\cos(k_2x_2)}{\sqrt{\cos^2(k_2x_2)+\cos^2(k_1x_1)}} \\ -\frac{\alpha}{\lambda}\frac{\cos(k_1x_1)}{\sqrt{\cos^2(k_2x_2)+\cos^2(k_1x_1)}} \\ 0 \end{pmatrix} \qquad (49)$$

and the space-like vector

$$\widehat{X}_I = \begin{pmatrix} 0 \\ \frac{\cos(k_1x_1)}{\sqrt{\cos^2(k_2x_2)+\cos^2(k_1x_1)}} \\ \frac{\cos(k_2x_2)}{\sqrt{\cos^2(k_2x_2)+\cos^2(k_1x_1)}} \\ 0 \end{pmatrix} \qquad (50)$$

A complication arises at points where λ goes from real to imaginary. In the latter case the two null eigenvectors of $G_{\mu\nu}$ give rise to two space-like vectors. The vector orthogonal to the vector that goes through from time-like to space-like, and thereafter gives the unit time-like vector, is

$$\begin{pmatrix} \frac{\alpha}{|\lambda|} \\ \frac{m^2}{|\lambda|}\cos(k_2x_2) \\ \frac{-m^2}{|\lambda|}\cos(k_1x_1) \\ 0 \end{pmatrix} \qquad (51)$$

This is proportional to ϕ_μ (imaginary field) which switches over in the same way. We now have two time-like two-planes as shown in figure 3. One notes that $\widehat{T}_I$ is obtained by a Lorentz boost in the plane of $(\widehat{T}_R, \widehat{X}_R)$ so that

$$\widehat{T}_I = \cosh(\theta)\widehat{T}_R + \sinh(\theta)\widehat{X}_R$$

therefore

$$\begin{aligned}\cosh\theta &= \frac{m^2}{\lambda}\sqrt{\cos^2(k_1x_1)+\cos^2(k_2x_2)}\\ \sinh\theta &= \frac{\alpha}{\lambda}\\ \tanh\theta &= \frac{\alpha}{m^2}\sqrt{\cos^2(k_1x_1)+\cos^2(k_2x_2)}\end{aligned} \tag{52}$$

In this case the intersection of the two time-like two-planes gives the common vector $\widehat{T}_I$ so that one can express both parts of $T_{\mu\nu}$ in terms of $\widehat{T}_I$ and $\widehat{X}_R$ boosted.

10.1. Derivation of the Time-like Eigenvector

Apart from terms in $g_{\mu\nu}$, $T_{\mu\nu}$ can be written as (see section 8)

$$\begin{aligned}&k_R\left[T_{\mu R}T_{\nu R}-X_{\mu R}X_{\nu R}\right]+m^2\phi_{\mu R}\phi_{\nu R}\\ &+k_I\left[T_{\mu I}T_{\nu I}-X_{\mu I}X_{\nu I}\right]+m^2\phi_{\mu I}\phi_{\nu I}\end{aligned} \tag{53}$$

Without changing the form of the expression one may apply a Lorentz boost to $(T_{\mu R}, X_{\mu R})$. Noting that $\widehat{X}_R$ is orthogonal to $\widehat{X}_I$ (46,50) we change our notation as follows

$$\begin{aligned}&\widehat{T}_{\mu I}\longrightarrow\widehat{T}_\mu\\ &\widehat{X}_{\mu I}\longrightarrow\widehat{Y}_\mu\\ &\widehat{X}_R(boosted)\longrightarrow\widehat{X}\end{aligned}$$

the relationship of the vectors is shown in figure 4. We can now write the expression (53) as

$$\begin{aligned}&k_R\left[\widehat{T}_\mu\widehat{T}_\nu-\widehat{X}_\mu\widehat{X}_\nu\right]+m^2\phi_{\mu R}\phi_{\nu R}\\ &+k_I\left[\widehat{T}_\mu\widehat{T}_\nu-\widehat{Y}_\mu\widehat{Y}_\nu\right]+m^2\phi_{\mu I}\phi_{\nu I}\end{aligned} \tag{54}$$

$\phi_{\mu R}$ is orthogonal to both $\widehat{T}_R$ and $\widehat{X}_R$ and proportional to $\widehat{X}_I\equiv\widehat{Y}_R$ (which we have now labelled $\widehat{Y}$). $\phi_{\mu I}$ is orthogonal to both $\widehat{T}_{\mu I}(\equiv\widehat{T}_\mu)$ and $\widehat{X}_{\mu I}(\equiv\widehat{Y}_\mu)$ and is now proportional to the boosted $\widehat{X}_R$, which is now $\widehat{X}$. We now have

$$\begin{aligned}&k_R\left[\widehat{T}_\mu\widehat{T}_\nu-\widehat{X}_\mu\widehat{X}_\nu\right]+m^2\widehat{Y}_\mu\widehat{Y}_\nu\times(const)\\ &+k_I\left[\widehat{T}_\mu\widehat{T}_\nu-\widehat{Y}_\mu\widehat{Y}_\nu\right]+m^2\widehat{X}_\mu\widehat{X}_\nu\times(const)\end{aligned} \tag{55}$$

One immediately sees that $\widehat{T}$ is the time-like eigen vector. The four velocities are then

$$\frac{dt}{d\tau}=\frac{m^2}{\lambda}\sqrt{\cos^2(k_1x_1)+\cos^2(k_2x_2)}$$

$$\frac{dx_1}{d\tau} = \frac{\alpha}{\lambda}\frac{\cos(k_2x_2)}{\sqrt{\cos^2(k_1x_1)+\cos^2(k_2x_2)}}$$
$$\frac{dx_2}{d\tau} = -\frac{\alpha}{\lambda}\frac{\cos(k_1x_1)}{\sqrt{\cos^2(k_1x_1)+\cos^2(k_2x_2)}} \tag{56}$$

with definitions as given previously in equation (47).

The eigenvalue belonging to the time-like eigenvector is easily calculated from the $G_{\mu\nu}$ and ϕ_μ previously given (see section 5.1). In this particular example the terms in $g_{\mu\nu}$ are equal to zero and therefore the overall eigenvalue will be

$$\begin{aligned} k_R + k_I = & \left[m^2k_0^2\left(cos^2(k_1x_1)+cos^2(k_2x_2)\right)\right] \\ + & \left[m^4\left(cos^2(k_1x_1)+cos^2(k_2x_2)\right) - m^2\left(k_1sin(k_1x_1)+k_2sin(k_2x_2)\right)^2\right] \end{aligned} \tag{57}$$

and $|k_1| = |k_2|$.

When λ becomes imaginary the four velocities can be read off from the new unit time-like vector (51) as

$$\frac{dt}{d\tau} = \frac{\alpha}{|\lambda|}$$
$$\frac{dx_1}{d\tau} = m^2\frac{\cos(k_2x_2)}{|\lambda|} \tag{58}$$
$$\frac{dx_2}{d\tau} = -m^2\frac{\cos(k_1x_1)}{|\lambda|}$$

The overall eigenvalue remains as given before in equation (57). The standing wave pattern which arises in the eigenvalue is shown in figure 5. For the purposes of the illustration we take $m = \hbar = c = 1$ and $k_1 = k_2 = 0.2$. Figure 6 shows the flow-lines in x_1, x_2, t around one of the minima in the eigenvalue.

11. Appendix

A general result for choosing tetrads in the case of the maxwellian field has been given in [10], [11] and [13]. In the case of current interest one needs to include the specification of the vector potential ϕ_μ by the tetrads. We therefore need to modify the choices made in [10], [11] and [13].

The vector field tensor, $G_{\mu\nu}$, is subjected to a duality rotation [7] giving an extremal field

$${}^{(\theta)}G_{\mu\nu} = G_{\mu\nu}cos(\theta) +^{*} G_{\mu\nu}sin(\theta)$$

One finds that

$$\left({}^{(\theta)}G_{\mu\nu}\right)\left({}^{*(\theta)}G^{\nu\lambda}\right) = g_\mu^\lambda\left[cos(2\theta)(\underline{E}\cdot\underline{B}) - sin(2\theta)\left(\frac{E^2-B^2}{2}\right)\right]$$

where $\underline{\mathrm{E}}$ and $\underline{\mathrm{B}}$ are the two 3-vectors specifying $G_{\mu\nu}$. Choosing

$$tan(2\theta) = \frac{2E\cdot B}{E^2-B^2}$$

gives

$$\left({}^{(\theta)}G_{\mu\nu}\right)\left({}^{*}(\theta)G^{\nu\lambda}\right) = 0 \tag{59}$$

One notes from [10] and [11], that the maxwellian part of the stress-energy-momentum tensor is unaltered by such a duality rotation. In terms of the extremal field and two arbitrary vectors one obtains a tetrad defining a time-like two-plane and an orthogonal space-like two-plane:

$$\begin{aligned}
U_\mu &= k\left({}^{(\theta)}G_{\mu\nu}\right)\theta^\nu \\
V_\mu &= \frac{1}{k}\left({}^{(\theta)}G_{\mu\nu}\right)U^\nu \\
W_\mu &= k\left({}^{*}\left({}^{(\theta)}G_{\mu\nu}\right)\right)\triangle^\nu \\
Z_\mu &= \frac{1}{k}\left({}^{*}\left({}^{(\theta)}G_{\mu\nu}\right)\right)W^\nu
\end{aligned} \tag{60}$$

θ^ν and $\triangle^\nu$ are two arbitrary vectors, in general, but will be chosen to be ϕ^ν for our purposes. k is the magnitude of the eigenvalues of the maxwellian part of the stress-energy-momentum tensor (its inclusion is not necessary for the arguments following but is helpful in normalisation.) The tetrads are not normalised as given. The extremal property (59) ensures that the two-planes defined by (U_μ, V_μ) and (W_μ, Z_μ) are orthogonal for any choice of θ^ν and $\triangle^\nu$. Using the anti-symmetry of ${}^{(\theta)}G_{\mu\nu}$ it is easy to show that:

(i) θ^ν is orthogonal to U^ν

(ii) V^ν is orthogonal to U^ν

A similar result will also apply to the other two-plane involving the duals of ${}^{(\theta)}G_{\mu\nu}$. The plane defined by (U_ν, V_ν) will be time-like (see the details in section 8). Choosing $\theta^\nu = \phi^\nu$ one has

(i) ϕ_ν time-like gives V^ν space-like and hence U^ν time-like

(ii) ϕ^ν space-like gives V^ν time0like and hence U^ν space-like

Using the orthogonality properties of θ^ν and $\triangle^\nu$ one finds that ϕ^ν has components in both two-planes

$$\phi^\nu = \frac{(\phi\cdot U)}{(U\cdot U)}U^\nu + \frac{(\phi\cdot Z)}{(Z\cdot Z)}Z^\nu$$

or

$$\phi^\nu = \frac{(\phi\cdot V)}{(V\cdot V)}V^\nu + \frac{(\phi\cdot W)}{(W\cdot W)}W^\nu$$

(This confirms the intuitively obvious result used in section 8).

12. Figure Captions

Figure 1 The maxwellian part of $T_{\mu\nu}$ gives a time-like two-plane $\left(\widehat{T}-\widehat{Z}\right)$ and the potential ϕ_μ is in the plane of $\left(\widehat{T}-\widehat{X}\right)$

Figure 2 The maxwellian part of $T_{\mu\nu}$ gives a time-like two-plane $\left(\widehat{T}-\widehat{Z}\right)$ but ϕ_μ is in the plane of $\left(\widehat{Z}-\widehat{Y}\right)$

Figure 3 The intersection of the two time-like two-planes gives the common vector $\widehat{T}_{Imaginary}$

Figure 4 The relationship between the vectors, $\widehat{T}_R, \widehat{T}, \widehat{X}, \widehat{X}_R, \widehat{Y}$.

Figure 5 The standing wave pattern, in the $x_1 - x_2$ plane, which arises in the eigenvalue for the illustrative example ($m = \hbar = c = 1$ and $k_1 = k_2 = 0.2$).

Figure 6 A set of flow lines of energy-momentum in the $x_1 - x_2 - t$ space-time for the illustrative example ($m = \hbar = c = 1$ and $k_1 = k_2 = 0.2$).

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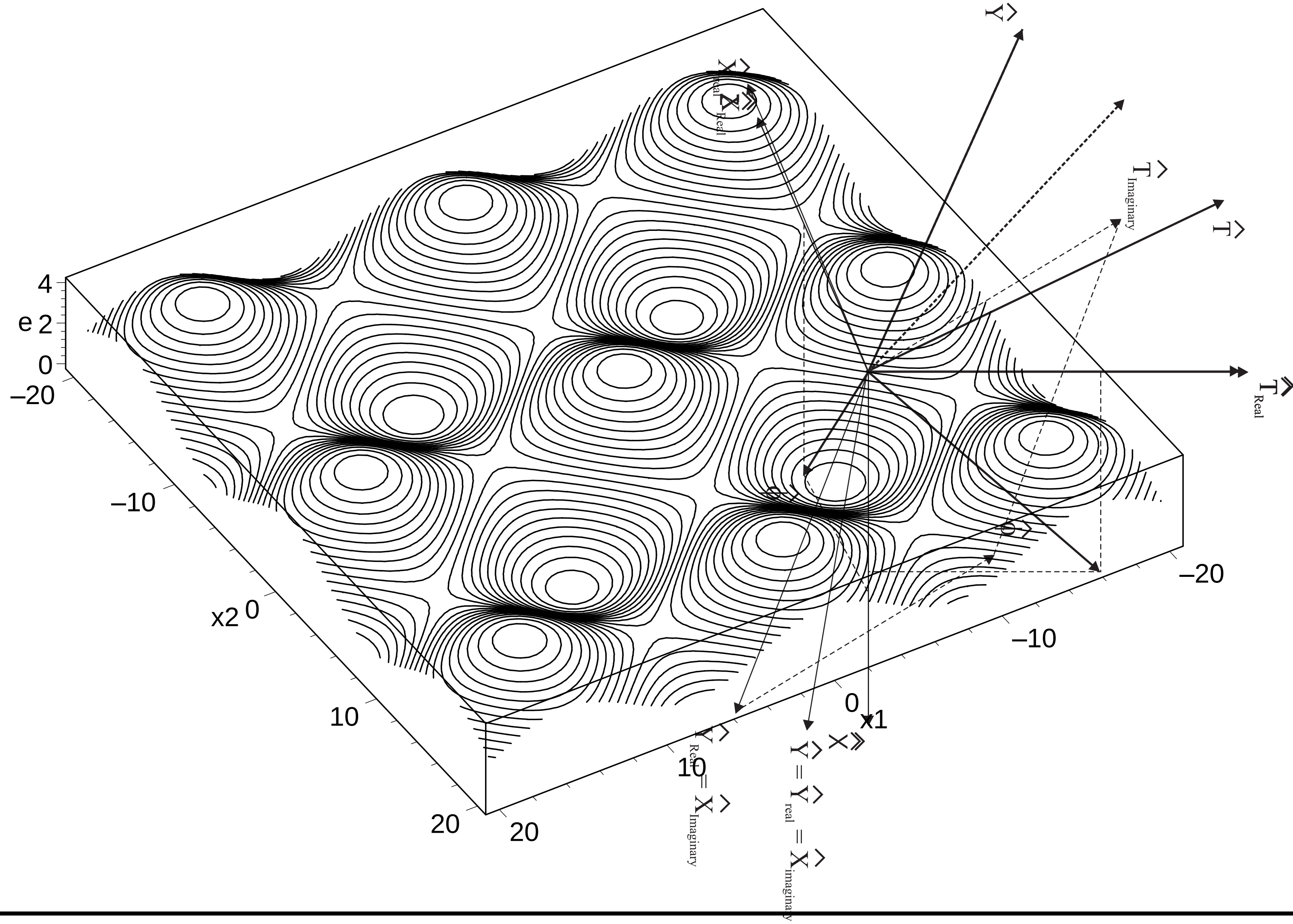

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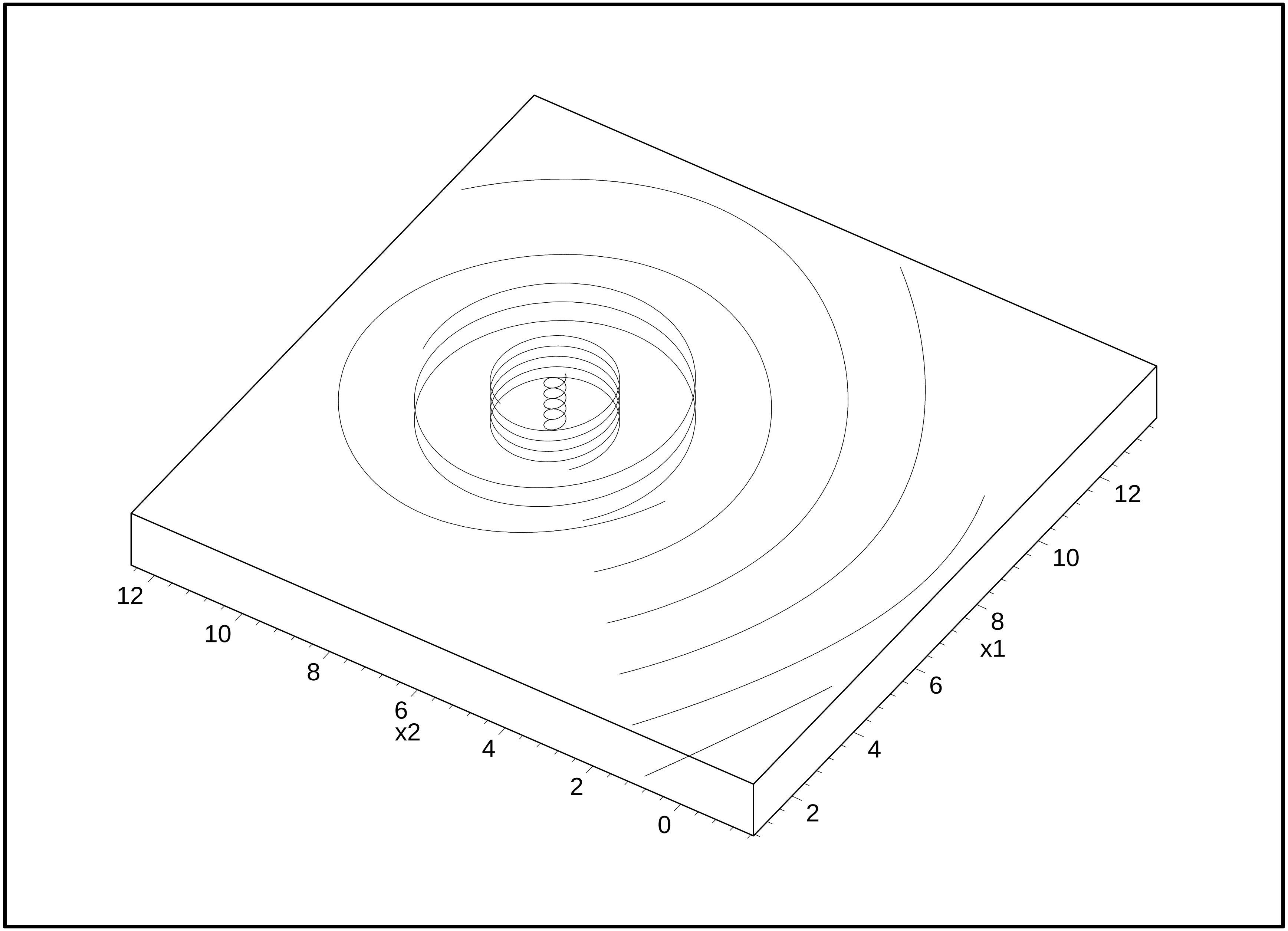
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